J. Fluid Mech. (1993), vol. 254, pp. 41–58

Hydrodynamic forces on a submerged circular cylinder undergoing large-amplitude motion

By G. X. WU

Department of Mechanical Engineering, University College London, Torrington Place, London WC1E 7JE, UK

(Received 4 March 1992 and in revised form 21 January 1993)

The hydrodynamic problem of a circular cylinder submerged below a free surface and undergoing large-amplitude oscillation is investigated based on the velocity potential theory. The body-surface boundary condition is satisfied on its instantaneous position while the free-surface condition is linearized. The solution is obtained by writing the potential in terms of the multipole expansion. Various interesting results associated with the circular cylinder are obtained.

1. Introduction

In the prediction of wave-induced motion of a marine structure, it is common to assume that the parameters under investigations are small. The result is the well-known linearized velocity potential theory which neglects all nonlinear terms from the equations and the boundary condition is imposed on the mean position. More recently attempts have been made to solve the exact nonlinear problem for an ideal fluid flow. Although significant progress has been made, the work in this area is still at its early stage and the results obtained so far are not entirely satisfactory. The major difficulty is the existence of the free surface, on which not only is the boundary condition nonlinear, but its position is unknown before the solution is found. Because of this difficulty, no analytical solution has been found even for the simplest geometry and it is unlikely that in the foreseeable future any such solutions can be found.

If an analytical solution beyond the linearized potential theory is desired, a more realistic approach is to retain the linearized free-surface condition but try to satisfy the exact rigid body-surface condition. Indeed, this not only achieves mathematical simplification but also has practical significance. For a deeply submerged body undergoing large-amplitude oscillation, its free-surface condition may still be linearized but its body-surface condition has to be satisfied on its exact position.

We shall consider here a submerged circular cylinder undergoing large-amplitude motion in water of infinite depth. If its submergence h is much larger than its radius a, the velocity potential on the free surface will be of order a/h provided that the vertical oscillation does not reduce the submergence significantly. Thus neglect of the product terms in the free-surface conditions will lead to an error of order $(a/h)^2$. The velocity potential for the circular cylinder can be written in terms of the well-known multipole expansion. The multipole expansion was initiated by Ursell (1949) who considered the linearized problem of a semicircular cylinder on the free surface. Ursell (1950) later used this method for a submerged circular cylinder, and it was extended by Ogilvie (1963) to include some nonlinear results. For finite water depth Yu & Ursell (1961) considered a semicircular cylinder on the free surface. Evans & Linton (1989) and Wu & Eatock Taylor (1990) also used the multipole expansion for a submerged circular cylinder in finite water depth when investigating the reduction of wave intensity and the second-order wave diffraction force respectively. However there is a significant difference when the multipole expansion is applied to the present problem. Based on the linearized theory, a cylinder oscillating with frequency ω only generates one wave with the same frequency. For the present problem, the cylinder will generate an infinite number of waves with frequencies $n\omega$ ($n = 1, 2, \ldots$). Correspondingly the procedure must be modified.

The investigation has two purposes other than as an analytical solution. Firstly, it will give some insight into the differences between the results obtained from the linearized body surface condition and the exact body surface condition. Secondly, the results obtained can be used to validate the computer programs based on this mathematical model which have been popular in the past few years (e.g. Lin & Yue 1990 and Ferrant 1990).

2. Governing equation

We consider the problem of a submerged circular cylinder of radius a undergoing sinusoidal oscillation with frequency ω. We define a coordinate system O, xz so that the origin is located on the undisturbed free surface and z points upwards. We also define a polar coordinate system (r, θ) so that the origin is fixed at the centre of the cylinder. These two systems are related by the following relationship:

$$x = r\sin\theta + \eta_1 \cos\alpha_1, \tag{1a}$$

$$z = r\cos\theta - (h - \eta_3 \cos\alpha_3), \tag{1b}$$

$$\alpha_j = \omega t + \gamma_j, \tag{1c}$$

where h is the mean submergence, η_1 and η_3 are the amplitudes of the horizontal motion and the vertical motion respectively; and γ_j ($j = 1, 3$) are the corresponding initial phases. It is assumed that at no stage will the cylinder emerge from the water, i.e. $\eta_3 + a < h$.

Based on the assumption of velocity potential theory, the potential Φ satisfies the following equations:

$$\nabla^2 \Phi = 0 \tag{2}$$

in the fluid domain;

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = 0 \tag{3a}$$

on the free surface S_F or $z = 0$, where g is the acceleration due to gravity; and

$$\frac{\partial \Phi}{\partial n} = -\omega(\eta_1 \sin\alpha_1\, n_1 + \eta_3 \sin\alpha_3\, n_3) \tag{3b}$$

on the body surface S_0 or $r = a$, where n is the inward normal of the body surface and $n_1 = -\sin\theta$ and $n_3 = -\cos\theta$ are its components in the x- and z-directions respectively. As discussed in the introduction, we have linearized the free-surface boundary condition but retained the exact body-surface condition. The radiation condition at $x = \pm\infty$ ensures that waves generated by the body propagate outwards.

We may define

$$\Phi = -\omega\eta_1 \operatorname{Re}(\phi_1 e^{i\gamma_1}) - \omega\eta_3 \operatorname{Re}(\phi_3 e^{i\gamma_3}), \tag{4}$$

in which ϕ_j satisfies (2), (3*a*), and

$$\frac{\partial \phi_1}{\partial r} = -\tfrac{1}{2}[\mathrm{e}^{\mathrm{i}(\omega t+\theta)} - \mathrm{e}^{\mathrm{i}(\omega t-\theta)}], \tag{5a}$$

$$\frac{\partial \phi_3}{\partial r} = -\tfrac{1}{2}\mathrm{i}\,[\mathrm{e}^{\mathrm{i}(\omega t+\theta)} + \mathrm{e}^{\mathrm{i}(\omega t-\theta)}] \tag{5b}$$

on the body surface. When the body-surface condition is linearized, ϕ_1 is due to the horizontal motion only and ϕ_3 due to the vertical motion only. As one would expect here, both ϕ_1 and ϕ_3 depend on the horizontal and vertical motions. Furthermore, when the body-surface condition is linearized, the time factor can be decoupled from the potential, or $\phi_j(x,z,t) = \psi_j(x,z)\,\mathrm{e}^{\mathrm{i}\omega t}$. But this does not apply in (5) because r is a function of time as seen from (1).

3. The multipole expansion

To solve the above problem, we write the potential in terms of the multipole expansion

$$\phi_j = \sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty} A_m^s\, a^m \left[\frac{\mathrm{e}^{\mathrm{i}m\theta+\mathrm{i}s\omega t}}{r^m} + f_m^s(r,\theta,t)\right] + \sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty} B_m^s\, a^m \left[\frac{\mathrm{e}^{-\mathrm{i}m\theta+\mathrm{i}s\omega t}}{r^m} + g_m^s(r,\theta,t)\right]. \tag{6}$$

The first terms in the square brackets are for the circular cylinder in an unbounded fluid domain while the second terms are introduced to satisfy the free-surface condition. Since when $|\theta| < \frac{1}{2}\pi$

$$\frac{\mathrm{e}^{\pm\mathrm{i}m\theta}}{r^m} = \frac{1}{(m-1)!}\int_0^\infty k^{m-1}\exp\left[-kr\,\mathrm{e}^{-(\pm)\mathrm{i}\theta}\right]\mathrm{d}k$$
$$= \frac{1}{(m-1)!}\int_0^\infty k^{m-1}\exp\left[-k(z+h)\pm\mathrm{i}kx + k\eta_3\cos\alpha_3 - (\pm)\,\mathrm{i}k\eta_1\cos\alpha_1\right]\mathrm{d}k \tag{7}$$

and (Abramowitz & Stegun 1965)

$$\exp\left[k\eta_3\cos\alpha_3\right] = \sum_{p=-\infty}^{\infty} I_p(k\eta_3)\,\mathrm{e}^{\mathrm{i}p\alpha_3}, \tag{8a}$$

$$\exp\left[(\pm)\,\mathrm{i}k\eta_1\cos\alpha_1\right] = \sum_{p=-\infty}^{\infty} (\pm\mathrm{i})^p J_p(k\eta_1)\,\mathrm{e}^{\mathrm{i}p\alpha_1}, \tag{8b}$$

where J_p and I_p are the Bessel functions and the modified Bessel functions respectively, we have

$$\frac{\mathrm{e}^{\pm\mathrm{i}m\theta+\mathrm{i}s\omega t}}{r^m} = \frac{1}{(m-1)!}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}(-1)^q(\pm\mathrm{i})^q\,\mathrm{e}^{\mathrm{i}p\alpha_3+\mathrm{i}q\alpha_1+\mathrm{i}s\omega t} \times \int_0^\infty k^{m-1}\,\mathrm{e}^{-k(z+h)\pm\mathrm{i}kx}\, I_p(k\eta_3)\, J_q(k\eta_1)\,\mathrm{d}k. \tag{9}$$

If we now write

$$f_m^s, g_m^s = \frac{1}{(m-1)!} \sum_{p=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} (-1)^q (\pm \mathrm{i})^q \, \mathrm{e}^{\mathrm{i}p\alpha_3 + \mathrm{i}q\alpha_1 + \mathrm{i}s\omega t} \times \int_0^{\infty} k^{m-1} A(k) \, \mathrm{e}^{k(z-h) \pm \mathrm{i}kx} I_p(k\eta_3) J_q(k\eta_1) \, \mathrm{d}k$$

and invoke the free-surface boundary condition in (3*a*), we obtain

$$A(k) = \frac{k+(p+q+s)^2\nu}{k-(p+q+s)^2\nu},$$

where $\nu = \omega^2/g$. The potential in (6) can then be written as

$$\begin{aligned}\phi_j = & \sum_{m=1}^{\infty} \sum_{s=-\infty}^{\infty} A_m^s a^m \left\{ \frac{\mathrm{e}^{\mathrm{i}m\theta + \mathrm{i}s\omega t}}{r^m} + \frac{1}{(m-1)!} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (-\mathrm{i})^q \, \mathrm{e}^{\mathrm{i}p\alpha_3 + \mathrm{i}q\alpha_1 + \mathrm{i}s\omega t} \right. \\ & \left. \times \int_L k^{m-1} \frac{k+(p+q+s)^2\nu}{k-(p+q+s)^2\nu} \mathrm{e}^{k(z-h)+\mathrm{i}kx} I_p(k\eta_3) J_q(k\eta_1) \, \mathrm{d}k \right\} \\ & + \sum_{m=1}^{\infty} \sum_{s=-\infty}^{\infty} B_m^s a^m \left\{ \frac{\mathrm{e}^{-\mathrm{i}m\theta + \mathrm{i}s\omega t}}{r^m} + \frac{1}{(m-1)!} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \mathrm{i}^q \, \mathrm{e}^{\mathrm{i}p\alpha_3 + \mathrm{i}q\alpha_1 + \mathrm{i}s\omega t} \right. \\ & \left. \times \int_L k^{m-1} \frac{k+(p+q+s)^2\nu}{k-(p+q+s)^2\nu} \mathrm{e}^{k(z-h)+\mathrm{i}kx} I_p(k\eta_3) J_q(k\eta_1) \, \mathrm{d}k \right\}. \end{aligned} \tag{10}$$

The integration route L is from zero to infinity. As the radiation condition only allows the outgoing wave, the integration route passes over those singularities with $p+q+s > 0$ and passes under those with $p+q+s < 0$.

To impose the body-surface condition we use (see (8))

$$\begin{aligned} \mathrm{e}^{k(z-h) \pm \mathrm{i}kx} &= \exp[-2kh + r\,\mathrm{e}^{\pm \mathrm{i}\theta} + k\eta_3 \cos\alpha_3 \pm \mathrm{i}k\eta_1 \cos\alpha_1] \\ &= \mathrm{e}^{-2kh} \sum_{n=0}^{\infty} \frac{k^n r^n \mathrm{e}^{\pm \mathrm{i}n\theta}}{n!} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (\pm \mathrm{i})^q I_p(k\eta_3) J_q(k\eta_1) \, \mathrm{e}^{\mathrm{i}p\alpha_3 + \mathrm{i}q\alpha_1}. \end{aligned} \tag{11}$$

Equation (10) becomes

$$\begin{aligned}\phi_j = & \sum_{m=1}^{\infty} \sum_{s=-\infty}^{\infty} A_m^s a^m \left\{ \frac{\mathrm{e}^{\mathrm{i}m\theta + \mathrm{i}s\omega t}}{r^m} + \frac{1}{(m-1)!} \sum_{n=0}^{\infty} \frac{r^n \mathrm{e}^{\mathrm{i}n\theta}}{n!} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{p_1=-\infty}^{\infty} \sum_{q_1=-\infty}^{\infty} \right. \\ & \left. \times (-\mathrm{i})^q \mathrm{i}^{q_1} \exp[\mathrm{i}p\alpha_3 + \mathrm{i}q\alpha_1 + \mathrm{i}p_1\alpha_3 + \mathrm{i}q_1\alpha_1 + \mathrm{i}s\omega t] F(m,n,p,q,p_1,q_1,s+p+q) \right\} \\ & + \sum_{m=1}^{\infty} \sum_{s=-\infty}^{\infty} B_m^s a^m \left\{ \frac{\mathrm{e}^{-\mathrm{i}m\theta + \mathrm{i}s\omega t}}{r^m} + \frac{1}{(m-1)!} \sum_{n=0}^{\infty} \frac{r^n \mathrm{e}^{-\mathrm{i}n\theta}}{n!} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{p_1=-\infty}^{\infty} \sum_{q_1=-\infty}^{\infty} \right. \\ & \left. \times (-\mathrm{i})^{q_1} \mathrm{i}^{q} \exp[\mathrm{i}p\alpha_3 + \mathrm{i}q\alpha_1 + \mathrm{i}p_1\alpha_3 + \mathrm{i}q_1\alpha_1 + \mathrm{i}s\omega t] F(m,n,p,q,p_1,q_1,s+p+q) \right\}, \end{aligned} \tag{12}$$

where

$$F(m,n,p,q,p_1,q_1,s) = \int_L k^{m+n-1} \frac{k+s^2\nu}{k-s^2\nu} \mathrm{e}^{-2kh} I_p(k\eta_3) J_q(k\eta_1) I_{p_1}(k\eta_3) J_{q_1}(k\eta_1) \, \mathrm{d}k. \tag{13}$$

Invoking (5), we obtain for ϕ_1

$$-\frac{m}{a}A^s_m+\sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}\sum_{u=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{p_1=-\infty}^{\infty}(-\mathrm{i})^q\mathrm{i}^{q_1}$$
$$\times\exp\left[\mathrm{i}(p+p_1)\gamma_3+\mathrm{i}(q+q_1)\gamma_1\right]F(m,n,p,q,p_1,q_1,u+p+q)\,A^u_n$$
$$=-\tfrac{1}{2}\delta(m-1)\,\delta(s-1), \quad (14a)$$

$$-\frac{m}{a}B^s_m+\sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}\sum_{u=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{p_1=-\infty}^{\infty}(-\mathrm{i})^{q_1}\mathrm{i}^{q}$$
$$\times\exp\left[\mathrm{i}(p+p_1)\gamma_3+\mathrm{i}(q+q_1)\gamma_1\right]F(m,n,p,q,p_1,q_1,u+p+q)\,B^u_n$$
$$=\tfrac{1}{2}\delta(m-1)\,\delta(s-1), \quad (14b)$$

where $q_1=s-p-q-p_1-u$, and $\delta(m)=1$ if $m=0$ and $\delta(m)=0$ otherwise. Similarly, we obtain for ϕ_3

$$-\frac{m}{a}A^s_m+\sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}\sum_{u=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{p_1=-\infty}^{\infty}(-\mathrm{i})^q\mathrm{i}^{q_1}$$
$$\times\exp\left[\mathrm{i}(p+p_1)\gamma_3+\mathrm{i}(q+q_1)\gamma_1\right]F(m,n,p,q,p_1,q_1,u+p+q)\,A^u_n$$
$$=-\tfrac{1}{2}\mathrm{i}\delta(m-1)\,\delta(s-1), \quad (15a)$$

$$-\frac{m}{a}B^s_m+\sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}\sum_{u=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{p_1=-\infty}^{\infty}(-\mathrm{i})^{q_1}\mathrm{i}^{q}$$
$$\times\exp\left[\mathrm{i}(p+p_1)\gamma_3+\mathrm{i}(q+q_1)\gamma_1\right]F(m,n,p,q,p_1,q_1,u+p+q)\,B^u_n$$
$$=-\tfrac{1}{2}\mathrm{i}\delta(m-1)\,\delta(s-1). \quad (15b)$$

Comparing (14) with (15), we find

$$A^s_m(3)=\mathrm{i}A^s_m(1), \qquad B^s_m(3)=-\mathrm{i}B^s_m(1), \quad (16a, b)$$

where j in $A^s_m(j)$ and $B^s_m(j)$ indicates that the coefficients correspond to ϕ_j.

4. The hydrodynamic force

The solution of (14) and (15) can be obtained by truncating the infinite series at a finite number, depending on the accuracy required. When the solution has been found, the hydrodynamic force can be obtained by integrating the pressure obtained from Bernoulli's equation over the body surface:

$$F_j=-\rho\int_{S_0}\left(\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi\right)n_j\,\mathrm{d}S \quad (17)$$

where ρ is the density of the fluid and the hydrostatic term gz has been neglected. F_1 in the above equation is the force in the horizontal direction and F_3 is the force in the vertical direction. The derivative with respect to time cannot be simply taken outside of the integral. In fact we should use

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_0}\Phi n_j\,\mathrm{d}S=\int_{S_0}\frac{\partial\Phi}{\partial t}n_j\,\mathrm{d}S+\int_{S_0}\left(\frac{\partial\Phi}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial\Phi}{\partial z}\frac{\partial z}{\partial t}\right)n_j\,\mathrm{d}S$$
$$=\int_{S_0}\frac{\partial\Phi}{\partial t}n_j\,\mathrm{d}S-\omega\int_{S_0}\left(\frac{\partial\Phi}{\partial x}\eta_1\sin\alpha_1+\frac{\partial\Phi}{\partial z}\eta_3\sin\alpha_3\right)n_j\,\mathrm{d}S, \quad (18)$$

where (1) has been used. If we further use Stokes theorem,

$$\int_{S_0}\frac{\partial\Phi}{\partial x_j}n_i\,\mathrm{d}S=\int_{S_0}\frac{\partial\Phi}{\partial x_i}n_j\,\mathrm{d}S, \quad (19)$$

where $(x_1, x_3) = (x, z)$, equation (18) may also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_0} \Phi n_j \,\mathrm{d}S = \int_{S_0} \frac{\partial \Phi}{\partial t} n_j \,\mathrm{d}S + \int_{S_0} \frac{\partial \Phi}{\partial x_j}\frac{\partial \Phi}{\partial n}\,\mathrm{d}S, \tag{20}$$

which is identical to equation (4.87) of Newman (1977) derived from the transport theorem. Substituting (18) into (17), we obtain

$$F_j = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_0} \rho \Phi n_j \,\mathrm{d}S$$
$$-\tfrac{1}{2}\rho \int_{S_0} \nabla(\Phi + \omega\eta_1 \sin\alpha_1 x + \omega\eta_3 \sin\alpha_3 z)\cdot\nabla(\Phi + \omega\eta_1 \sin\alpha_1 x + \omega\eta_3 \sin\alpha_3 z)\, n_j \,\mathrm{d}S. \tag{21}$$

On the body surface, the expression for ϕ_j can be simplified by substituting (14) or (15) into (12). We have

$$\phi_1 = 2\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty} [A_m^s(1)\,\mathrm{e}^{\mathrm{i}m\theta} + B_m^s(1)\,\mathrm{e}^{-\mathrm{i}m\theta}]\,\mathrm{e}^{\mathrm{i}s\omega t} - \mathrm{i}a\sin\theta\,\mathrm{e}^{\mathrm{i}\omega t} + f(t) \tag{22a}$$

and

$$\phi_3 = 2\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty} [A_m^s(3)\,\mathrm{e}^{\mathrm{i}m\theta} + B_m^s(3)\,\mathrm{e}^{-\mathrm{i}m\theta}]\,\mathrm{e}^{\mathrm{i}s\omega t} - \mathrm{i}a\cos\theta\,\mathrm{e}^{\mathrm{i}\omega t} + g(t), \tag{22b}$$

where $f(t)$ and $g(t)$ arise from the term $n = 0$ in (12). Substituting (22) into (4), we obtain

$$\Phi = 2\,\mathrm{Re}\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty} (C_m^s\,\mathrm{e}^{\mathrm{i}m\theta} + D_m^s\,\mathrm{e}^{-\mathrm{i}m\theta})\,\mathrm{e}^{\mathrm{i}s\omega t} - \omega a\eta_1 \sin\theta \sin\alpha_1 - \omega a\eta_3 \cos\theta\sin\alpha_3, \tag{23}$$

where $f(t)$ and $g(t)$ have been dropped since they make no contribution to the force, and

$$C_m^s = -\omega\eta_1 A_m^s(1)\,\mathrm{e}^{\mathrm{i}\gamma_1} - \omega\eta_3 A_m^s(3)\,\mathrm{e}^{\mathrm{i}\gamma_3} = -\omega(\eta_1\mathrm{e}^{\mathrm{i}\gamma_1} + \mathrm{i}\eta_3\mathrm{e}^{\mathrm{i}\gamma_3})\,A_m^s(1), \tag{24a}$$
$$D_m^s = -\omega\eta_1 B_m^s(1)\,\mathrm{e}^{\mathrm{i}\gamma_1} - \omega\eta_3 B_m^s(3)\,\mathrm{e}^{\mathrm{i}\gamma_3} = -\omega(\eta_1\mathrm{e}^{\mathrm{i}\gamma_1} - \mathrm{i}\eta_3\mathrm{e}^{\mathrm{i}\gamma_3})\,B_m^s(1). \tag{24b}$$

From (21), (3b) and (23), we obtain

$$F_j = -\rho a\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} \Phi n_j \,\mathrm{d}\theta$$
$$-\frac{\rho}{2a}\int_0^{2\pi} \frac{\partial}{\partial\theta}(\Phi + \omega\eta_1\sin\alpha_1 x + \omega\eta_3 \sin\alpha_3 z)\frac{\partial}{\partial\theta}(\Phi + \omega\eta_1\sin\alpha_1 x + \omega\eta_3\sin\alpha_3 z)\,n_j\,\mathrm{d}\theta$$
$$= -\rho\pi\omega^2 a^2\eta_j\cos\alpha_j - 2\rho a\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\int_0^{2\pi}\left\{\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty}(C_m^s\,\mathrm{e}^{\mathrm{i}m\theta} + D_m^s\,\mathrm{e}^{-\mathrm{i}m\theta})\,\mathrm{e}^{\mathrm{i}s\omega t}\, n_j\,\mathrm{d}\theta\right.$$
$$-\frac{\rho}{a}\mathrm{Re}\int_0^{2\pi}\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty}\mathrm{i}m(C_m^s\,\mathrm{e}^{\mathrm{i}m\theta} - D_m^s\,\mathrm{e}^{-\mathrm{i}m\theta})\,\mathrm{e}^{\mathrm{i}s\omega t}$$
$$\times\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty}\mathrm{i}m[(C_m^s\,\mathrm{e}^{\mathrm{i}m\theta} - D_m^s\,\mathrm{e}^{-\mathrm{i}m\theta})\,\mathrm{e}^{\mathrm{i}s\omega t} + (-\bar{C}_m^s\,\mathrm{e}^{-\mathrm{i}m\theta} + \bar{D}_m^s\,\mathrm{e}^{\mathrm{i}m\theta})\,\mathrm{e}^{-\mathrm{i}s\omega t}]\,n_j\,\mathrm{d}\theta$$
$$= -\rho\pi\omega^2 a^2\eta_j\cos\alpha_j - 2\rho a\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\int_0^{2\pi}\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty}(C_m^s\,\mathrm{e}^{\mathrm{i}m\theta} + D_m^s\,\mathrm{e}^{-\mathrm{i}m\theta})\,\mathrm{e}^{\mathrm{i}s\omega t}\,n_j\,\mathrm{d}\theta$$
$$-\frac{\rho}{a}\mathrm{Re}\int_0^{2\pi}\sum_{m=1}^{\infty}\sum_{m_1=1}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{s_1=-\infty}^{\infty}\left\{[mm_1(C_m^s D_{m_1}^{s_1}\,\mathrm{e}^{\mathrm{i}(m-m_1)\theta} + \Delta_\mu^\sigma X_{\mu_1}^{\sigma_1}\,\mathrm{e}^{-\mathrm{i}(m-m_1)\theta})]\,\mathrm{e}^{\mathrm{i}(s+s_1)\omega t}\right.$$
$$\left. + [mm_1(C_m^s\bar{C}_{m_1}^{s_1}\,\mathrm{e}^{\mathrm{i}(m-m_1)\theta} + D_m^s\bar{D}_{m_1}^{s_1}\,\mathrm{e}^{-\mathrm{i}(m-m_1)\theta})]\,\mathrm{e}^{\mathrm{i}(s-s_1)\omega t}\right\} n_j\,\mathrm{d}\theta,$$

where the overbar indicates the complex conjugate. This gives

$$F_1 = -\rho\pi\omega^2 a^2 \eta_1 \cos\alpha_1 + \rho\pi\,\mathrm{Re}\Big\{2a \sum_{s=-\infty}^{\infty} s\omega(-C_1^s + D_1^s)\,\mathrm{e}^{\mathrm{i}s\omega t}$$
$$+\frac{\mathrm{i}}{a}\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{s_1=-\infty}^{\infty}[m(m+1)(C_{m+1}^s D_m^{s_1} - D_{m+1}^s C_m^{s_1} - C_m^s D_{m+1}^{s_1} + D_m^s C_{m+1}^{s_1})\,\mathrm{e}^{\mathrm{i}(s+s_1)\omega t}$$
$$+m(m+1)(C_{m+1}^s \bar{C}_m^{s_1} - D_{m+1}^s \bar{D}_m^{s_1} - C_m^s \bar{C}_{m+1}^{s_1} + D_m^s \bar{D}_{m+1}^{s_1})]\,\mathrm{e}^{\mathrm{i}(s-s_1)\omega t}\Big\}, \qquad (25a)$$

$$F_3 = -\rho\pi\omega^2 a^2 \eta_3 \cos\alpha_3 + \rho\pi\,\mathrm{Re}\Big\{2a \sum_{s=-\infty}^{\infty} \mathrm{i}s\omega(C_1^s + D_1^s)\,\mathrm{e}^{\mathrm{i}s\omega t}$$
$$+\frac{1}{a}\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{s_1=-\infty}^{\infty}[m(m+1)(C_{m+1}^s D_m^{s_1} + D_{m+1}^s C_m^{s_1} + C_m^s D_{m+1}^{s_1} + D_m^s C_{m+1}^{s_1})\,\mathrm{e}^{\mathrm{i}(s+s_1)\omega t}$$
$$+m(m+1)(C_{m+1}^s \bar{C}_m^{s_1} + D_{m+1}^s \bar{D}_m^{s_1} + C_m^s \bar{C}_{m+1}^{s_1} + D_m^s \bar{D}_{m+1}^{s_1})]\,\mathrm{e}^{\mathrm{i}(s-s_1)\omega t}\Big\}. \qquad (25b)$$

We may also write the force as

$$F_j = \mathrm{Re}\left[\sum_{s=-\infty}^{\infty} F_j(s)\,\mathrm{e}^{\mathrm{i}st}\right], \qquad (26)$$

where

$$F_1(s) = -\tfrac{1}{2}\rho\pi\omega^2 a^2 \eta_1\,\mathrm{e}^{\mathrm{i}s\gamma_1}[\delta(s-1)+\delta(s+1)] + \rho\pi\Big\{2as\omega(-C_1^s + D_1^s)$$
$$+\frac{\mathrm{i}}{a}\sum_{m=1}^{\infty}\sum_{s_1=-\infty}^{\infty}[m(m+1)(C_{m+1}^{s-s_1} D_m^{s_1} - D_{m+1}^{s-s_1} C_m^{s_1}$$
$$-C_m^{s-s_1} D_{m+1}^{s_1} + D_m^{s-s_1} C_{m+1}^{s_1} + C_{m+1}^{s+s_1}\bar{C}_m^{s_1} - D_{m+1}^{s+s_1}\bar{D}_m^{s_1} - C_m^{s+s_1}\bar{C}_{m+1}^{s_1}$$
$$+D_m^{s+s_1}\bar{D}_{m+1}^{s_1})]\Big\}, \qquad (27a)$$

$$F_3(s) = -\tfrac{1}{2}\rho\pi\omega^2 a^2 \eta_3\,\mathrm{e}^{\mathrm{i}s\gamma_3}[\delta(s-1)+\delta(s+1)] + \rho\pi\Big\{2a\mathrm{i}s\omega(C_1^s + D_1^s)$$
$$+\frac{1}{a}\sum_{m=1}^{\infty}\sum_{s_1=-\infty}^{\infty}[m(m+1)(C_{m+1}^{s-s_1} D_m^{s_1} + D_{m+1}^{s-s_1} C_m^{s_1}$$
$$+C_m^{s-s_1} D_{m+1}^{s_1} + D_m^{s-s_1} C_{m+1}^{s_1} + C_{m+1}^{s+s_1}\bar{C}_m^{s_1} + D_{m+1}^{s+s_1}\bar{D}_m^{s_1} + C_m^{s+s_1}\bar{C}_{m+1}^{s_1}$$
$$+D_m^{s+s_1}\bar{D}_{m+1}^{s_1})]\Big\}. \qquad (27b)$$

The horizontal steady force may also be related to the asymptotic expansion of the potential at infinity. We add (18) to (20) and substitute the result into (17). This gives

$$F_j = -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_0}\Phi n_j\,\mathrm{d}S + \frac{\rho}{2}\int_{S_0}\left[\frac{\partial\Phi}{\partial x_j}\frac{\partial\Phi}{\partial n} - \nabla(\Phi + \omega\eta_1\sin\alpha_1 x + \omega\eta_3\sin\alpha_3 z)\cdot\nabla\Phi n_j\right]\mathrm{d}S.$$

Using the equation derived by Ogilvie & Tuck (1969),

$$\int_{S_0}\nabla\psi\cdot\nabla\phi\, n_j\,\mathrm{d}S = \int_{S_0}\frac{\partial^2\psi}{\partial n\,\partial x_j}\phi\,\mathrm{d}S,$$

if $\partial\psi/\partial n = 0$ on S_0, we have

$$F_j = -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_0}\Phi n_j\,\mathrm{d}S + \frac{\rho}{2}\int_{S_0}\left[\frac{\partial\Phi}{\partial x_j}\frac{\partial\Phi}{\partial n} - \frac{\partial^2\Phi}{\partial n\,\partial x_j}\Phi\right]\mathrm{d}S$$
$$= -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_0}\Phi n_j\,\mathrm{d}S - \frac{\rho}{2}\int_{S_F+S_\infty}\left[\frac{\partial\Phi}{\partial x_j}\frac{\partial\Phi}{\partial n} - \frac{\partial^2\Phi}{\partial n\,\partial x_j}\Phi\right]\mathrm{d}S, \qquad (28)$$

where S_∞ comprises two vertical lines at $x = \pm\infty$ respectively. Using (3a), we obtain for the horizontal force

$$F_1 = -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_0}\Phi n_1\,\mathrm{d}S - \frac{\rho}{2g}\int_{S_F}(\Phi_x\Phi_{tt} - \Phi_{xtt}\Phi)\,\mathrm{d}S - \frac{\rho}{2}\int_{S_\infty}\left(\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial n} - \frac{\partial^2\Phi}{\partial n\,\partial x}\Phi\right)\mathrm{d}S$$
$$= -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_0}\Phi n_1\,\mathrm{d}S - \frac{\rho}{2g}\frac{\partial}{\partial t}\int_{S_F}(\Phi_x\Phi_t - \Phi_{xt}\Phi)\,\mathrm{d}S - \frac{\rho}{2}\int_{S_\infty}\left(\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial n} - \frac{\partial^2\Phi}{\partial n\,\partial x}\Phi\right)\mathrm{d}S. \tag{29}$$

Let $T = 2\pi/\omega$, we then have

$$F_1(0) = \frac{1}{T}\int_0^T F_1\,\mathrm{d}t = -\frac{\rho}{2T}\int_0^T\int_{S_\infty}\left(\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial n} - \frac{\partial^2\Phi}{\partial n\,\partial x}\Phi\right)\mathrm{d}S\,\mathrm{d}t. \tag{30}$$

It may seem the derivation of (30) is unnecessary, since this result is well known from momentum conservation (e.g. Mei 1982, eq. 10.11, p. 368). But care is needed here. The derivation from momentum conservation is based on the fact that all boundary conditions are satisfied exactly. Here, we have used the exact body-surface condition and linearized free-surface condition. Although, (30) turns out to be identical to that derived from momentum conservation, that is not guaranteed when inconsistent conditions are used. Indeed, a well-known example is what was called by Eggers (1979) Gadd's paradox the in wave resistance problem of a floating body advancing in otherwise calm water. Because of the inconsistency of the body-surface and the free-surface conditions, the integration of the pressure over the body surface does not give the same force as that obtained from energy conservation. Thus, the derivation of (30) here is not trivial.

Let $x \to +\infty$ in (10). Taking into account (4) and (24), we have

$$\Phi = 2\pi\,\mathrm{Re}\Bigg\{\mathrm{i}\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty}C_m^s\frac{a^m}{(m-1)!}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}(-\mathrm{i})^q\,\mathrm{e}^{\mathrm{i}p\alpha_3+\mathrm{i}q\alpha_1+\mathrm{i}s\omega t}$$
$$\times[1-\mathrm{sgn}\,(p+q+s)]\,(p+q+s)^{2m}\,\nu^m\,I_p[(p+q+s)^2\nu\eta_3]\,J_q[(p+q+s)^2\nu\eta_1]$$
$$\times\exp\,[(p+q+s)^2\nu(z-h)+\mathrm{i}(p+q+s)^2\nu x]$$
$$+\mathrm{i}\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty}D_m^s\frac{a^m}{(m-1)!}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\mathrm{i}^q\,\mathrm{e}^{\mathrm{i}p\alpha_3+\mathrm{i}q\alpha_1+\mathrm{i}s\omega t}$$
$$\times[-1-\mathrm{sgn}\,(p+q+s)]\,(p+q+s)^{2m}\,\nu^m\,I_p[(p+q+s)^2\nu\eta_3]\,J_q[(p+q+s)^2\nu\eta_1]$$
$$\times\exp\,[(p+q+s)^2\nu(z-h)-\mathrm{i}(p+q+s)^2\nu x]\Bigg\},\quad x\to+\infty, \tag{31a}$$

$$\Phi = 2\pi\,\mathrm{Re}\Bigg\{\mathrm{i}\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty}C_m^s\frac{a^m}{(m-1)!}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}(-\mathrm{i})^q\,\mathrm{e}^{\mathrm{i}p\alpha_3+\mathrm{i}q\alpha_1+\mathrm{i}s\omega t}$$
$$\times[-1-\mathrm{sgn}\,(p+q+s)]\,(p+q+s)^{2m}\,\nu^m\,I_p[(p+q+s)^2\nu\eta_3]\,J_q[(p+q+s)^2\nu\eta_1]$$
$$\times\exp\,[(p+q+s)^2\nu(z-h)+\mathrm{i}(p+q+s)^2\nu x]$$
$$+\mathrm{i}\sum_{m=1}^{\infty}\sum_{s=-\infty}^{\infty}D_m^s\frac{a^m}{(m-1)!}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\mathrm{i}^q\,\mathrm{e}^{\mathrm{i}p\alpha_3+\mathrm{i}q\alpha_1+\mathrm{i}s\omega t}$$
$$\times[1-\mathrm{sgn}\,(p+q+s)]\,(p+q+s)^{2m}\,\nu^m\,I_p[(p+q+s)^2\nu\eta_3]\,J_q[(p+q+s)^2\nu\eta_1]$$
$$\times\exp\,[(p+q+s)^2\nu(z-h)-\mathrm{i}(p+q+s)^2\nu x]\Bigg\},\quad x\to-\infty, \tag{31b}$$

where $\operatorname{sgn}(s) = 1$ if $s > 0$ and $\operatorname{sgn}(s) = -1$ if $s < 0$. Equation (31) may also be written as

$$\Phi = \operatorname{Re}\left\{\sum_{s=1}^{\infty}[f_1(s)\,e^{s^2\nu z+is^2\nu x-is\omega t}+f_2(s)\,e^{s^2\nu z-is^2\nu x+is\omega t}]\right\}$$

$$= \operatorname{Re}\left\{\sum_{s=1}^{\infty}[\bar{f}_1(s)+f_2(s)]\,e^{s^2\nu z-is^2x+is\omega t}\right\}, \quad x\to+\infty, \tag{32a}$$

$$\Phi = \operatorname{Re}\left\{\sum_{s=1}^{\infty}[g_1(s)\,e^{s^2\nu z+is^2\nu x+is\omega t}+g_2(s)\,e^{s^2\nu z-is^2\nu x-is\omega t}]\right\}, \quad x\to-\infty$$

$$= \operatorname{Re}\left\{\sum_{s=1}^{\infty}[g_1(s)+\bar{g}_2(s)]\,e^{s^2\nu z+is^2x+is\omega t}\right\}, \quad x\to-\infty, \tag{32b}$$

where

$$f_1(s) = 4\pi i\,e^{-s^2\nu h}\sum_{m=1}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}C_m^{-s-p-q}\frac{(s^2\nu a)^m}{(m-1)!}I_p(s^2\nu\eta_3)\,J_q(s^2\nu\eta_1)(-i)^q\,e^{ip\gamma_3+iq\gamma_1}, \tag{33a}$$

$$f_2(s) = -4\pi i\,e^{-s^2\nu h}\sum_{m=1}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}D_m^{s-p-q}\frac{(s^2\nu a)^m}{(m-1)!}I_p(s^2\nu\eta_3)\,J_q(s^2\nu\eta_1)\,i^q\,e^{ip\gamma_3+iq\gamma_1}, \tag{33b}$$

$$g_1(s) = -4\pi i\,e^{-s^2\nu h}\sum_{m=1}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}C_m^{s-p-q}\frac{(s^2\nu a)^m}{(m-1)!}I_p(s^2\nu\eta_3)\,J_q(s^2\nu\eta_1)(-i)^q\,e^{ip\gamma_3+iq\gamma_1}, \tag{33c}$$

$$g_2(s) = 4\pi i\,e^{-s^2\nu h}\sum_{m=1}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}D_m^{-s-p-q}\frac{(s^2\nu a)^m}{(m-1)!}I_p(s^2\nu\eta_3)\,J_q(s^2\nu\eta_1)\,i^q\,e^{ip\gamma_3+iq\gamma_1}. \tag{33d}$$

Substituting (32) into (30), we obtain

$$F_1(0) = -\frac{\rho\nu}{4}\sum_{s=1}^{\infty}s^2[|f(s)|^2-|g(s)|^2], \tag{34}$$

where $\quad f = \bar{f}_1+f_2, \quad g = \bar{g}_1+g_2.$

5. Some special cases

We shall consider some special cases, which will not only give some very interesting results but also may be used as a partial check on the present formulation.

5.1. *The linear problem*

When η_1 and η_3 are sufficiently small, we only need to keep their linear terms. Because of (4), we can take $\eta_1 = \eta_3 = 0$ in the expansion of ϕ_j. Since $J_0(0) = I_0(0) = 1$ and $J_n(0) = I_n(0) = 0$ if $|n| > 0$, we only need to retain the terms of $p = p_1 = q = q_1 = 0$ in (14) and (15). The equation for ϕ_1 becomes

$$-\frac{m}{a}A_m^s+\sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}F(m,n,0,0,0,0,s)\,A_n^s = -\tfrac{1}{2}\delta(m-1)\,\delta(s-1), \tag{35a}$$

$$-\frac{m}{a}B_m^s+\sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}F(m,n,0,0,0,0,s)\,B_n^s = \tfrac{1}{2}\delta(m-1)\,\delta(s-1). \tag{35b}$$

We notice that the equations for $s = 0, \pm 1, \ldots$ respectively are completely uncoupled. In particular, (35) gives $A_m^s = B_m^s = 0$ if $s \neq 1$ and it becomes the governing equation for the linear problem when $s = 1$. The comparison of (35a) and (35b) also shows that $A^s_m = -B^s_m$. This suggests ϕ_1 in (10) is antisymmetric about $\theta = 0$ or $x = 0$, which is well known.

The same analysis can also be applied to ϕ_3 by replacing the right-hand sides of (35) with those in (15). The difference is that in this case $A^s_m = B^s_m$ and ϕ_3 is symmetric.

5.2. *Purely vertical motion*

When $\eta_1 = 0$, we only need to consider ϕ_3 because of (4). For the reason discussed above we only need to retain the terms of $q = q_1 = 0$ in (15). We have

$$-\frac{m}{a}A^s_m + \sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}\sum_{u=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} e^{i(s-u)\gamma_3} F(m,n,p,0,s-u-p,0,u+p)\,A^u_n = -\tfrac{1}{2}i\delta(m-1)\,\delta(s-1), \quad (36a)$$

$$-\frac{m}{a}B^s_m + \sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}\sum_{u=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} e^{i(s-u)\gamma_3} F(m,n,p,0,s-u-p,0,u+p)\,B^u_n = -\tfrac{1}{2}i\delta(m-1)\,\delta(s-1). \quad (36b)$$

These two equations are identical, which implies $A^s_m = B^s_m$. Thus ϕ_3 in (10) is symmetric about $\theta = 0$ and $F_1 = 0$ in (25), which is hardly surprising for a symmetric cylinder undergoing purely vertical motion.

5.3. *Purely horizontal motion*

When $\eta_3 = 0$, we only need to retain the terms of $p = p_1 = 0$ in (14). We have

$$-\frac{m}{a}A^s_m + \sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}\sum_{u=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} F(m,n,0,q,0,s-q-u,u+q) \times(-1)^q\, i^{s-u}\, e^{i(s-u)} A^u_n = -\tfrac{1}{2}\delta(m-1)\,\delta(s-1), \quad (37a)$$

$$-\frac{m}{a}B^s_m + \sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}\sum_{u=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} F(m,n,0,q,0,s-q-u,u+q) \times(-1)^{q+s+u}\, i^{s-u}\, e^{i(s-u)} B^u_n = \tfrac{1}{2}\delta(m-1)\,\delta(s-1). \quad (37b)$$

It is evident here that $A^s_m = (-1)^s B^s_m$. Substituting this into (25), we obtain $F_1(s) = 0$ when s is even and $F_3(s) = 0$ when s is odd. In particular $F_1(0) = 0$ reflects the physical fact that a cylinder symmetric about $x = 0$ which oscillates sinusoidally in the horizontal direction does not give steady force in the horizontal direction.

5.4. *Circular motion*

When $\eta_1 = \eta_3 = \eta$ and $\gamma_3 = \gamma_1 \pm \frac{1}{2}\pi = \gamma$, the cylinder moves in a circular path with the centre at $(0, -h)$. In particular, when the positive sign is taken, the motion is clockwise and when the negative sign is taken it is counterclockwise. From (24), we notice that the former corresponds to $C^s_m = 0$ and the later corresponds to $D^s_m = 0$. For the linear problem such a result leads to the conclusion that a circular cylinder moving in a circular path only radiates the wave in one direction (Ogilvie 1963). Take clockwise motion as an example. We have $f_1(s) = g_1(s) = 0$ in (33a, c) because $C^s_m = 0$. For the linear problem, we only keep the term corresponding to $p = q = 0$ and $D^s_m = 0$ if

$s \neq 1$ (see §5.1), which further gives $g_2(s) = 0$ in (33d). Thus we only have $f_2(s)$ in (32), which implies that there is a wave only at $x = +\infty$. Similarly if the motion is counterclockwise, we only need to retain $g_1(s)$ and it only radiates the wave to $x = -\infty$. Further interesting results may be obtained from (34). For clockwise motion the steady horizontal force is always negative while for counterclockwise motion, the force is always positive. The magnitudes of the forces in both cases are the same.

For the problem satisfying the exact body-surface condition, many of the above conclusions are no longer valid. We shall give a detailed discussion in Appendix A where it shows that for circular motion the multipole expansion has a much simpler form. Here we shall confirm an apparent physical result based on the present mathematical model (exact body-surface boundary condition and linearized free-surface condition): when the circular motion changes from clockwise to counterclockwise only the horizontal forces (not only the steady force) change sign while the vertical forces remain the same. We may write (14b) as

$$-\frac{m}{a}B^s_m + \sum_{n=1}^{\infty} \frac{a^{m+n-1}}{(m-1)!\,(n-1)!} \sum_{u=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{p_1=-\infty}^{\infty} F(m,n,p,q,p_1,q_1,u+p+q) \times (-1)^q \mathrm{e}^{\mathrm{i}(p+p_1)\gamma_3} B^u_n = \tfrac{1}{2}\delta(m-1)\,\delta(s-1) \quad (38a)$$

for clockwise motion ($\gamma_1 = \gamma_3 - \frac{1}{2}\pi$). We do not need to consider A^s_m because of (24a). Similar we may write

$$-\frac{m}{a}A^s_m + \sum_{n=1}^{\infty} \frac{a^{m+n-1}}{(m-1)!\,(n-1)!} \sum_{u=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{p_1=-\infty}^{\infty} F(m,n,p,q,p_1,q_1,u+p+q) \times (-1)^q \mathrm{e}^{\mathrm{i}(p+p_1)\gamma_3} A^u_n = -\tfrac{1}{2}\delta(m-1)\,\delta(s-1) \quad (38b)$$

for counterclockwise motion ($\gamma_1 = \gamma_3 + \frac{1}{2}\pi$) and disregard B^s_m. Comparing (38a) and (38b), we find the solutions of these two equations only differ by a sign. From (24), we obtain

$$D^s_m \text{ (clockwise)} = C^s_m \text{ (counterclockwise)} = -2\mathrm{i}\omega\eta\,\mathrm{e}^{\mathrm{i}\gamma_3} A^s_m(1), \quad (39a)$$

$$D^s_m \text{ (counterclockwise)} = C^s_m \text{ (clockwise)} = 0, \quad (39b)$$

where the A^s_m are the solution of (38b). Substituting (39) into (27), we obtain

$$F_1 \text{ (clockwise)} = -F_1 \text{ (counterclockwise)}, \quad (40a)$$

$$F_3 \text{ (clockwise)} = F_3 \text{ (counterclockwise)}. \quad (40b)$$

In fact (40) is not limited to the cylinder undergoing circular motion only. The relationships are valid for a motion and its mirror image about $x = 0$, which is evident on physical ground. Mathematically, the mirror image means that γ_1 is replaced with $\gamma_1 + \pi$, while there are no restrictions on η_1, η_3, γ_1 and γ_3. As above, the comparison of these two cases will give

$$A^s_m(1)\,|_{\gamma_1+\pi} = -B^s_m(1)\,|_{\gamma_1}, \qquad B^s_m(1)\,|_{\gamma_1+\pi} = -A^s_m(\gamma_1)\,|_{\gamma_1}. \quad (41a, b)$$

Using the above equations in (24) then the result in (27), we obtain

$$F_1(\gamma_1+\pi) = -F_1(\gamma_1), \qquad F_3(\gamma_1-\pi) = F_3(\gamma_1). \quad (42a, b)$$

Before we discuss the numerical results, it is also interesting to confirm mathematically another evident physical point: the magnitudes of the forces in (27)

(a)

η_3/a	$c_3(0)$	$c_3(1)$	$c_3(2)$	$c_3(3)$	$c_3(4)$	$c_3(5)$	$c_3(6)$	$c_3(7)$
0	0	1.1114	0	0	0	0	0	0
0.20	0.0039	1.1122	0.0173	0.0011	0.0001	0	0	0
0.40	0.0081	1.1148	0.0353	0.0044	0.0005	0	0	0
0.60	0.0127	1.1193	0.0545	0.0103	0.0016	0.0002	0	0
0.80	0.0181	1.1262	0.0761	0.0192	0.0040	0.0007	0.0001	0
1.00	0.0247	1.1361	0.1011	0.0320	0.0084	0.0018	0.0003	0.0001
1.25	0.0360	1.1545	0.1399	0.0554	0.0182	0.0049	0.0012	0.0003
1.50	0.0531	1.1838	0.1927	0.0913	0.0356	0.0114	0.0034	0.0012
1.75	0.0838	1.2359	0.2709	1.1464	0.0640	0.0224	0.0080	0.0049

(b)

η_3/a	$c_3(0)$	$c_3(1)$	$c_3(2)$	$c_3(3)$	$c_3(4)$	$c_3(5)$	$c_3(6)$	$c_3(7)$
0.00	0	0.8781	0	0	0	0	0	0
0.20	−0.0049	0.8774	0.0100	0.0005	0	0	0	0
0.40	−0.0099	0.8754	0.0203	0.0021	0.0002	0	0	0
0.60	−0.0151	0.8720	0.0311	0.0048	0.0006	0.0001	0	0
0.80	−0.0205	0.8672	0.0427	0.0088	0.0015	0.0003	0	0
1.00	−0.0262	0.8608	0.0554	0.0143	0.0031	0.0006	0.0001	0
1.25	−0.0340	0.8505	0.0734	0.0237	0.0065	0.0017	0.0004	0.0001
1.50	−0.0424	0.8378	0.0943	0.0365	0.0121	0.0038	0.0012	0.0004
1.75	−0.0510	0.8231	0.1192	0.0541	0.0214	0.0082	0.0032	0.0012

TABLE 1. Purely vertical motion with $h = 3a$. (a) $\nu a = 0.1$, (b) $\nu a = 1.0$

depend only on the difference of the initial phases $\gamma_3-\gamma_1$ rather than them individually. Taking (14a) as an example we have, by noticing $q_1 = s-p-p_1-q-u$,

$$-\frac{m}{a}A^s_m+\sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}\sum_{u=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{p_1=-\infty}^{\infty}F(m,n,p,q,p_1,q_1,u+p+q)$$
$$\times(-\mathrm{i})^q\,\mathrm{i}^{q_1}\,\mathrm{e}^{\mathrm{i}(p+p_1)(\gamma_3-\gamma_1)}A^u_n\,\mathrm{e}^{\mathrm{i}(s-u)\gamma_1} = -\tfrac{1}{2}\delta(m-1)\,\delta(s-1).$$

This equation shows that $A^s_m\exp[-\mathrm{i}(s-1)\gamma_1]$ depends only on $\gamma_3-\gamma_1$. A similar conclusion applies to $B^s_m(1)\exp[-\mathrm{i}(s-1)\gamma_1]$. We now write (24) as

$$C^s_m = -\omega[\eta_1+\mathrm{i}\eta_3\,\mathrm{e}^{\mathrm{i}(\gamma_3-\gamma_1)}]\,A^s_m\,\mathrm{e}^{-\mathrm{i}(s-1)\gamma_1}\,\mathrm{e}^{\mathrm{i}s\gamma_1}, \qquad (43a)$$
$$D^s_m = -\omega[\eta_1-\mathrm{i}\eta_3\,\mathrm{e}^{\mathrm{i}(\gamma_3-\gamma_1)}]\,B^s_m\,\mathrm{e}^{-\mathrm{i}(s-1)\gamma_1}\,\mathrm{e}^{\mathrm{i}s\gamma_1}. \qquad (43b)$$

Using these results in (25), we find that γ_1 only affects the phases of the forces but not their magnitudes. This conclusion may have been easily obtained by replacing t with $t-\gamma_1/\omega$ in (1). The above exercise is relevant to the beginning of §5.

6. Numerical results

The numerical solution is obtained by truncating the infinite series in (1) at $m = M$ and $s = S$. Although no attempt is made to rigorously prove the convergence of the series at $M\rightarrow\infty$ and $S\rightarrow\infty$, the numerical investigations indicate that it converges very rapidly. For the results obtained below, we have taken $M = 5$ and $S = 10$. The Bessel functions have been calculated using their integral representations, as using the recurrence relations may lead to less accurate results when the order is high. Computations are made on a DECstation 5000. Priority is given to accuracy at the expense of computer time, since the program is not for general geometries.

(*a*)

η_1/a	$c_3(0)$	$c_1(1)$	$c_3(2)$	$c_1(3)$	$c_3(4)$	$c_1(5)$	$c_3(6)$	$c_1(7)$
0.00	0	1.1114	0	0	0	0	0	0
0.20	0.0039	1.1115	0.0142	0.0007	0	0	0	0
0.40	0.0079	1.1120	0.0285	0.0028	0.0003	0	0	0
0.60	0.0120	1.1127	0.0430	0.0064	0.0009	0.0001	0	0
0.80	0.0163	1.1136	0.0577	0.0113	0.0021	0.0003	0.0001	0
1.00	0.0208	1.1144	0.0728	0.0177	0.0040	0.0008	0.0002	0
1.25	0.0269	1.1150	0.0922	0.0275	0.0076	0.0019	0.0005	0.0001
1.50	0.0333	1.1147	0.1118	0.0392	0.0129	0.0039	0.0011	0.0003
1.75	0.0400	1.1130	0.1314	0.0526	0.0198	0.0675	0.0022	0.0007

(*b*)

η_1/a	$c_3(0)$	$c_1(1)$	$c_3(2)$	$c_1(3)$	$c_3(4)$	$c_1(5)$	$c_3(6)$	$c_1(7)$
0	0	0.8781	0	0	0	0	0	0
0.20	−0.0049	0.8784	0.0031	0.0001	0	0	0	0
0.40	−0.0098	0.8793	0.0060	0.0006	0	0	0	0
0.60	−0.0146	0.8807	0.0086	0.0012	0.0001	0	0	0
0.80	−0.0193	0.8828	0.0108	0.0021	0.0003	0	0	0
1.00	−0.0239	0.8856	0.0125	0.0031	0.0005	0.0001	0	0
1.25	−0.0293	0.8899	0.0138	0.0045	0.0010	0.0001	0	0
1.50	−0.0343	0.8952	0.0141	0.0058	0.0015	0.0003	0	0
1.75	−0.0387	0.9011	0.0134	0.0070	0.0020	0.0004	0.0001	0

TABLE 2. Purely horizontal motion with $h = 2a$. (*a*) $\nu a = 0.1$, (*b*) $\nu a = 1.0$

We may write (26) as

$$F_j = \mathrm{Re}\left\{F_j(0) + \sum_{s=1}^{\infty} [F_j(s) + \bar{F}_j(-s)]\,\mathrm{e}^{is\omega t}\right\}, \tag{44}$$

where F_j are calculated from (27), and equation (34) is used as a partial check. Table 1 gives some results for the cylinder undergoing purely vertical motion. The $c_j(s)$ are the non-dimensional force amplitudes and are defined as

$$c_3(0) = \mathrm{Re}\,[F_3(0)]/\rho\omega^2\pi a^2\eta_3 \quad \text{and} \quad c_j(s) = |F_j(s) + F_j(-s)|/\rho\omega^2\pi a^2\eta_3 \quad \text{if} \quad s > 0.$$

As discussed in §5.2, $c_1(s) = 0$ in this case and is therefore omitted from the table. Table 2 gives the results for the cylinder undergoing purely horizontal motion. The $c_j(s)$ are defined as

$$c_3(0) = \mathrm{Re}\,[F_3(0)]/\rho\omega^2\pi a^2\eta_1 \quad \text{and} \quad c_j(s) = |F_j(s) + F_j(-s)|/\rho\omega^2\pi a^2\eta_1.$$

Note that $c_1(2s) = c_2(2s+1) = 0$ ($s = 0, 1, \ldots$), as discussed in §5.3. For the linearized problem it is well known the $|F_3|$ due to the vertical motion is the same as $|F_1|$ due to the horizontal motion, which correspond to $\eta_3 = 0$ in table 1 and $\eta_1 = 0$ in table 2 respectively. When the amplitude of oscillation increases, this identity is no longer valid. From both tables 1 and 2, we see that the forces are dominated by that obtained from the linearized theory. In fact if we examined the assumption of the present formulation more carefully, the results should have been expected. For a cylinder in an unbounded fluid domain, the linear solution can be exact if the coordinate system fixed in space is replaced with that fixed on the cylinder. In particular, the force obtained from the linearized theory is identical to that obtained from the solution satisfying the exact body-surface condition, which only contains the term $F_j(1)$ in (26). All other

(*a*)

η/a	$c_1(0)$	$c_1(1)$	$c_1(2)$	$c_1(3)$	$c_1(4)$	$c_1(5)$	$c_1(6)$	$c_1(7)$
0	0	0.8988	0	0	0	0	0	0
0.20	−0.0140	0.8971	0.0123	0.0006	0	0	0	0
0.40	−0.0280	0.8923	0.0252	0.0027	0.0003	0	0	0
0.60	−0.0419	0.8843	0.0392	0.0062	0.0009	0.0001	0	0
0.80	−0.0558	0.8734	0.0548	0.0117	0.0023	0.0004	0.0001	0
1.00	−0.0694	0.8599	0.0724	0.0195	0.0049	0.0011	0.0002	0
1.25	−0.0853	0.8399	0.0978	0.0333	0.0105	0.0030	0.0007	0.0001
1.50	−0.0984	0.8173	0.1270	0.0527	0.0199	0.0066	0.0019	0.0004
1.75	−0.1055	0.7930	0.1593	0.0779	0.0334	0.0119	0.0033	0.0006

(*b*)

η/a	$c_3(0)$	$c_3(1)$	$c_3(2)$	$c_3(3)$	$c_3(4)$	$c_3(5)$	$c_3(6)$	$c_3(7)$
0	0	0.8988	0	0	0	0	0	0
0.20	−0.0013	0.8975	0.0123	0.0006	0	0	0	0
0.40	−0.0034	0.8934	0.0250	0.0026	0.0003	0	0	0
0.60	−0.0070	0.8859	0.0386	0.0062	0.0009	0.0001	0	0
0.80	−0.0130	0.8741	0.0538	0.0116	0.0023	0.0004	0	0
1.00	−0.0226	0.8566	0.0717	0.0194	0.0049	0.0011	0.0002	0
1.25	−0.0413	0.8237	0.0999	0.0341	0.0107	0.0030	0.0007	0.0002
1.50	−0.0698	0.7753	0.1372	0.0560	0.0206	0.0067	0.0019	0.0005
1.75	−0.1073	0.7132	0.1827	0.0845	0.0344	0.0119	0.0035	0.0013

TABLE 3. Circular motion with $h = 3a$ and $\nu a = 0.5$. (*a*) Horizontal force, (*b*) vertical force

terms in (26) are due to the free-surface effect. As it is assumed that the disturbance on the free surface is small and linearization can still be applied, the results in the tables are therefore not too surprising.

For vertical motion when $\eta_3/a > 1$, the nonlinear contribution increases rapidly and begins to show its presence. This is mainly because the motion in this case alters submergence substantially during the oscillation.

Table 3 gives the results for clockwise circular motion. One notable feature in this case is that there is a steady horizontal force. But the force is still dominated by that obtained from the linearized theory and the other components only begin to show their presence at large amplitude.

7. Conclusions

(i) A cylinder oscillating at frequency ω and with large amplitude will in general generate an infinite number of waves with frequencies $n\omega$ ($n = 1, 2, \ldots$) and propagating in both directions.

(ii) The purely horizontal motion of a submerged circular cylinder generates vertical forces with frequencies $2n\omega$ and horizontal forces with frequencies $(2n+1)\omega$ ($n = 0, 1, 2, \ldots$).

(iii) The large-amplitude circular motion of a submerged circular cylinder generates waves propagating in both directions, which differ from the well-known conclusion for small-amplitude circular motion (fully linearized theory).

(iv) Further, from Appendix B, a submerged circular cylinder undergoing large-amplitude motion reflects an incoming wave, which is also different from the results of the linearized theory.

(v) If the motion of the cylinder is not sinusoidal but still periodic, the method in this paper can still be used. However, the coefficients of the Fourier series in (8) may not be written in terms of the Bessel functions. Instead they may have to be calculated numerically. Consequently, J_p and I_p in those related equations may have to be replaced by numerical results.

The financial support by Cray Research (UK) Ltd is gratefully acknowledged.

Appendix A. Multipole expansion for the circular motion

For the circular motion discussed in §5.4, the multipole expansion can also be simplified. Without losing generality, we assume $\gamma = 0$. Equation (3 *b*) becomes

$$\partial\Phi/\partial r = -\omega\eta \sin[\omega t - (\pm)\theta], \tag{A 1}$$

where the signs + and − correspond to clockwise and counterclockwise motions respectively. We may define

$$\Phi = -\omega\eta\,\mathrm{Re}(\phi) \tag{A 2}$$

where

$$\partial\phi/\partial r = -\mathrm{i}\,\mathrm{e}^{\mathrm{i}[\omega t - (\pm)\theta]}. \tag{A 3}$$

ϕ can still be written in terms of (6), but instead of using (8), we use

$$\exp(k\eta\,\mathrm{e}^{\pm \mathrm{i}\alpha}) = \sum_{p=0}^{\infty} \frac{(k\eta)^p\,\mathrm{e}^{\pm \mathrm{i}p\alpha}}{p!}. \tag{A 4}$$

We notice that $\alpha = \omega t$ for clockwise motion and $\alpha = -\omega t$ for counterclockwise motion. Equation (10) becomes

$$\phi = \sum_{m=1}^{\infty} \sum_{s=-\infty}^{\infty} A_m^s a^m \left\{ \frac{\mathrm{e}^{\mathrm{i}m\theta + \mathrm{i}s\omega t}}{r^m} + \frac{1}{(m-1)!} \sum_{p=0}^{\infty} \mathrm{e}^{\mathrm{i}p\omega t + \mathrm{i}s\omega t} \frac{\eta^p}{p!} \times \int_L k^{m+p-1} \frac{k + (s+p)^2\nu}{k - (s+p)^2\nu} \mathrm{e}^{k(z-h) + \mathrm{i}kx}\,\mathrm{d}k \right\} \tag{A 5a}$$

for counterclockwise motion, and

$$\phi = \sum_{m=1}^{\infty} \sum_{s=-\infty}^{\infty} B_m^s a^m \left\{ \frac{\mathrm{e}^{-\mathrm{i}m\theta + \mathrm{i}s\omega t}}{r^m} + \frac{1}{(m-1)!} \sum_{p=0}^{\infty} \mathrm{e}^{\mathrm{i}p\omega t + \mathrm{i}s\omega t} \frac{\eta^p}{p!} \times \int_L k^{m+p-1} \frac{k + (s+p)^2\nu}{k - (s+p)^2\nu} \mathrm{e}^{k(z-h) + \mathrm{i}kx}\,\mathrm{d}k \right\} \tag{A 5b}$$

for clockwise motion. Similarly, we may write (11) as

$$\mathrm{e}^{k(z-h) \pm \mathrm{i}kx} = \mathrm{e}^{-2kh} \sum_{n=0}^{\infty} \frac{k^n r^n\,\mathrm{e}^{\pm \mathrm{i}n\theta}}{n!} \sum_{p=0}^{\infty} \frac{k^p\,\eta^p\,\mathrm{e}^{\pm \mathrm{i}p\alpha}}{p!}. \tag{A 6}$$

Equations (14) become

$$-\frac{m}{a} A_m^s + \sum_{n=1}^{\infty} \frac{a^{m+n-1}}{(m-1)!\,(n-1)!} \sum_{p=0}^{\infty} \sum_{p_1=0}^{\infty} \frac{\eta^{p+p_1}}{p!\,p_1!} A_n^{s-p+p_1} \times \int_L k^{m+n+p+p_1-1} \frac{k + (s+p_1)^2\nu}{k - (s+p_1)^2\nu} \mathrm{e}^{-2kh}\,\mathrm{d}k = -\mathrm{i}\delta(m-1)\,\delta(s-1), \tag{A 7a}$$

$$B_m^s = 0 \tag{A 7b}$$

for counterclockwise motion, and

$$A^s_m = 0, \tag{A 8a}$$

$$-\frac{m}{a}B^s_m + \sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}\sum_{p=0}^{\infty}\sum_{p_1=0}^{\infty}\frac{\eta^{p+p_1}}{p!\,p_1!}B_n^{s-p+p_1}$$
$$\times\int_L k^{m+n+p+p_1-1}\frac{k+(s+p_1)^2\nu}{k-(s+p_1)^2\nu}\mathrm{e}^{-2kh}\,\mathrm{d}k = -\mathrm{i}\delta(m-1)\,\delta(s-1), \tag{A 8b}$$

for clockwise motion. Comparing (A 7*a*) with (A 8*b*), we have

$$B^s_m\ (\text{clockwise}) = A^s_m\ (\text{counterclockwise}) \tag{A 9}$$

which has been shown in §5.4 (equations (36)).

Let $x\rightarrow\pm\infty$ and, in view of (A 9), (A 5*a*) for counterclockwise motion becomes

$$\phi = \sum_{s=1}^{\infty} f(-s)\exp(s^2\nu z+\mathrm{i}s^2\nu x-\mathrm{i}s\omega t), \quad x\rightarrow+\infty, \tag{A 10a}$$

$$\phi = -\sum_{s=1}^{\infty} f(s)\exp(s^2\nu z+\mathrm{i}s^2\nu x+\mathrm{i}s\omega t), \quad x\rightarrow-\infty, \tag{A 10b}$$

and (A 5*b*) for clockwise motion becomes

$$\phi = -\sum_{s=1}^{\infty} f(s)\exp(s^2\nu z-\mathrm{i}s^2\nu x+\mathrm{i}s\omega t), \quad x\rightarrow+\infty, \tag{A 11a}$$

$$\phi = \sum_{s=1}^{\infty} f(-s)\exp(s^2\nu z-\mathrm{i}s^2\nu x-\mathrm{i}s\omega t), \quad x\rightarrow-\infty, \tag{A 11b}$$

where

$$f(s) = 4\pi\mathrm{i}\sum_{m=1}^{\infty}\sum_{p=0}^{\infty} A^{s-p}_m\frac{a^m}{(m-1)!}\frac{\eta^p}{p!}(s^2\nu)^{m+p}\,\mathrm{e}^{-s^2\nu h}. \tag{A 12}$$

From the above results, there is no evidence to suggest that $f(-s) = 0$. Thus we are unable to conclude that the circular motion generates waves only in one direction as in the fully linearized theory. Indeed in the numerical calculation for the results in table 3, it is observed that $f(-s)$ is not zero.

Appendix B. Wave diffraction

For a deeply submerged cylinder, a large-amplitude motion is unlikely to be generated by surface waves. The body-surface condition in the diffraction problem can normally be imposed on the mean position. Suppose that the large-amplitude motion is generated by other causes and the body-surface condition has to be imposed on the instantaneous position. The wave diffraction will then be affected by the motion of the cylinder.

The incident potential Φ_i due to a regular incoming wave may be written as

$$\Phi_i = \mathrm{Re}\,(\phi_i) = \mathrm{Re}\left[-\frac{g}{\mathrm{i}\omega}\mathrm{e}^{\nu z\pm\mathrm{i}\nu x+\mathrm{i}\omega t}\right] \tag{B 1}$$

in which + sign corresponds to a wave from the right ($x = +\infty$) and − to a wave from the left ($x = -\infty$). If we write the diffraction potential as $\Phi_d = \mathrm{Re}\,(\phi_d)$, we have the body-surface condition

$$\partial\phi_d/\partial n = -\partial\phi_i/\partial n. \tag{B 2}$$

The multipole expansion for ϕ_d is the same as that in (10) but the coefficients are obtained from the following equation:

$$-\frac{m}{a}A_m^s+\sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}\sum_{u=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{p_1=-\infty}^{\infty}(-\mathrm{i})^q\,\mathrm{i}^{q_1}$$
$$\times\,\mathrm{e}^{\mathrm{i}(p+p_1)\gamma_3+\mathrm{i}(q+q_1)}F(m,n,p,q,p_1,q_1,u+p+q)\,A_m^u$$
$$=\frac{\nu g}{\mathrm{i}\omega}\mathrm{e}^{-\nu h}\frac{(\nu a)^{m-1}}{(m-1)!}\sum_{p=-\infty}^{\infty}\mathrm{i}^{s-p-1}I_p(k\eta_3)\,J_{s-p-1}(k\eta_1)\,\mathrm{e}^{\mathrm{i}p\gamma_3+\mathrm{i}(s-p-1)\gamma_1}, \quad \text{(B 3a)}$$

$$B_m^s=0, \quad \text{(B 3b)}$$

if the wave is from the right and

$$A_m^s=0, \quad \text{(B 4a)}$$

$$-\frac{m}{a}B_m^s+\sum_{n=1}^{\infty}\frac{a^{m+n-1}}{(m-1)!\,(n-1)!}\sum_{u=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{p_1=-\infty}^{\infty}(-\mathrm{i})^{q_1}\,\mathrm{i}^{q}$$
$$\times\,\mathrm{e}^{\mathrm{i}(p+p_1)\gamma_3+\mathrm{i}(q+q_1)}F(m,n,p,q,p_1,q_1,u+p+q)\,B_m^u$$
$$=\frac{\nu g}{\mathrm{i}\omega}\mathrm{e}^{-\nu h}\frac{(\nu a)^{m-1}}{(m-1)!}\sum_{p=-\infty}^{\infty}(-\mathrm{i})^{s-p-1}I_p(k\eta_3)\,J_{s-p-1}(k\eta_1)\,\mathrm{e}^{\mathrm{i}p\gamma_3+\mathrm{i}(s-p-1)\gamma_1} \quad \text{(B 4b)}$$

if the wave is from the left, where $q_1=s-p-q-p_1-u$. For the fully linearized problem, a well-known result for a circular cylinder is that it does not reflect (Dean 1948; Ursell 1950). But it is not evident here that we can draw such a conclusion since there is no explicit indication in (B 3) or (B 4) that we only need to retain the terms with $p+q+s>0$ in (10). One may speculate that since the zero reflection of the linearized theory is valid for any submergence it should not be changed by the oscillation of the cylinder. But there is a fundamental difference between these two cases. For the linearized theory, the whole problem becomes steady if the time factor $\exp(\mathrm{i}\omega t)$ is taken out. Although the zero reflection is valid for any submergence, the submergence is a constant once given and does not vary with time. For the present problem, the submergence does vary with time and so the above reasoning does not apply to this case.

REFERENCES

ABRAMOWITZ, M. & STEGUN, M. 1965 *Handbook of Mathematical Functions*. Dover.

DEAN, W. R. 1948 On the reflection of surface waves by a circular cylinder. *Proc. Camb. Phil. Soc.* **44**, 483–491.

EGGERS, K. W. H. 1979 A method for assessing numerical solution to the Neumann–Kelvin problem. *Proc. Workshop on Ship Wave Resistance Computations, Maryland, USA*. David Taylor Naval Ship Research and Development Center.

EVANS, D. V. & LINTON, C. M. 1989 Active devices for the reduction of wave intensity. *Appl. Ocean Res.* **11**, 26–32.

FERRANT, P. 1990 A coupled time and frequency approach for nonlinear wave radiation. *Eighteenth Symp. on Naval Hydrodynamics, University of Michigan*, pp. 67–83. National Academy Press.

LIN, W. M. & YUE, D. 1990 Numerical solutions for large-amplitude ship motions in the time domain. *Eighteenth Symp. on Naval Hydrodynamics, University of Michigan*, pp. 41–65. National Academy Press.

MEI, C. C. 1982 *The Applied Dynamics of Ocean Surface Waves*. Wiley-Interscience.

NEWMAN, J. 1977 *Marine Hydrodynamics*. MIT Press.

OGILVIE, T. F. 1963 First and second order forces on a cylinder submerged under the free surface. *J. Fluid Mech.* **16**, 451–472.

OGILVIE, T. F. & TUCK, E. O. 1969 A rational strip theory for ship motions. Part 1. *Rep.* 012. Department of Naval Architecture and Marine Engineering, University of Michigan.

URSELL, F. 1949 On the heaving motion of a circular cylinder on the surface of fluid. *Q. J. Mech. Appl. Maths* 2, 218–231.

URSELL, F. 1950 Surface waves on deep water in the presence of a submerged circular cylinder. *Proc. Camb. Phil. Soc.* **46**, 141–152.

WU, G. X. & EATOCK TAYLOR, R. 1990 The second order diffraction force on a horizontal cylinder in finite water depth. *Appl. Ocean Res.* 12, 106–111.

YU, Y. S. & URSELL, F. 1961 Surface waves generated by an oscillating circular cylinder on water of finite depth: theory and experiment. *J. Fluid Mech.* **11**, 529–549.